A condition of cooperation. Games on network

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Natural selection is often regarded as a result of severe competition. Defect seems beneficial for a single individual in many cases. However, cooperation is observed in many levels of biological systems ranging from single cells to animals, including human society. We have yet known that in unstructured populations, evolution favors defectors over cooperators. On the other hand, there have been much interest on evolutionary games[1,2] on structured population and on graphs[3–16]. Structures of biological systems and societies of animals can be taken as networks. They discover that network structures determine results of the games. Together with the recent interest of complex networks[17,18], many researchers investigate real network structures. Recently even economists study firms' transactions structure[19]. Seminal work[11] derives the condition of favoring cooperation for evolutionary games on networks, that is, benefit divided by cost, b/c, exceeds average degree, $\langle k \rangle$. Although this condition has been believed so far[20], we find the condition is $b/c > \langle k_{nn} \rangle$ instead. $\langle k_{nn} \rangle$ is the mean nearest neighbor degree. Our condition enables us to compare how network structure enhances cooperation across different kinds of networks. Regular network favors most, scale free network least. On ideal scale free networks, cooperation is unfeasible. We could say that $\langle k \rangle$ is the degree of itself, while $\langle k_{nn} \rangle$ is that of others. One of the most interesting points in

network theory is that results depend not only on itself but also on others'. In evolutionary games on network, we find the same characteristic.

There are only two kinds of agent: cooperator and defector. A cooperator pays a cost, c, for each neighbor to receive a benefit, b. On the other hand, a defector pays no cost and its neighbor does not receive any benefit. We also assume $b > c$. The payoff of the game can be described by the following payoff matrix

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \end{array} \tag{1}$$

In deterministic game, D-D is the unique Nash equilibrium and will be taken. Although cooperation by all the agents raises payoff of them, it is difficult to maintain cooperation because for a single agent defect is always beneficial than cooperation. Thus cooperation will collapse. In an unstructured population, where all the agents interact each other, defectors tend to have higher average gain than cooperators, so that cooperators would extinct as a result of natural selection. This is true for deterministic replicator equation[21,22] and stochastic game of finite population[23]. In the model of present paper, a network structure is introduced. Agent occupies each vertex of the network. They play games only with the agents on adjacent vertices. If there are l neighbors around the cooperator regardless of cooperators or defectors, he pays cost, lc. In addition, if j of them are cooperators he is benefited jb from the cooperators. His fitness from the game is $jb - lc$. If the defector has i cooperators around him, his fitness is ib. The game proceeds as follows. For each time step, randomly chosen one agent on the network dies. Then adjacent agents compete for the empty vertex proportional to their fitness. The fitness of each agent is given by the constant term, baseline fitness, plus the payoff of the game. When the payoff of the game is relatively smaller than the baseline fitness, we call this as weak selection. The idea of the weak selection is that there are many factors determining the whole fitness, the game in consideration is only one of the factors. Reproduction of strategy can be taken as generic and cultural. The former is biological selection, the latter is social phenomena.

The interest of the present paper comes from evolutionary biology and complex networks. We need to explain a few things about complex networks. Degree, k, is the number of edges the vertex has. If all the vertices have the same degree, the network is called regular network. If not, it is called non-regular network. $\langle k \rangle$ is the mean degree. $\langle k_{nn} \rangle = \langle k^2 \rangle / \langle k \rangle$ is the nearest neighbor mean degree. For instance, if you choose one vertex randomly, the mean degree of which is $\langle k \rangle$. If you look at vertices linked to the randomly chosen one, the mean degree of which is not $\langle k \rangle$ but $\langle k_{nn} \rangle$. The fact that $\langle k \rangle \neq \langle k_{nn} \rangle$ plays an important role in complex network theory. In our point of view many interesting results[24,25] comes from $\langle k \rangle \neq \langle k_{nn} \rangle$. The intuitive reason why they differ is that vertices are likely to be linked to the vertices with larger degree. We need to explain in what respect we can say a network favors cooperation. First we prepare a network where all the vertices are occupied by only defectors, then we replace one of them by a single cooperator, then we start the evolutionary games until either when all the vertices are occupied by only defectors or only cooperators. We iterate the same games and have the probability that only cooperators occupy the vertices. This probability is called fixation probability. If selection neither favors nor opposes cooperation the probability is $1/N$, N is the size of networks. If the fixation probability is larger than $1/N$, we say that the network structure favors cooperation. Thus the condition $b/c > \langle k_{nn} \rangle$ is like a threshold between network favors cooperation or opposes.

We derive the condition by mean field approximation. Mean field approximation is an useful method which replaces something by mean value to obtain analytical solution, which has been used especially in Physics. In mean field picture, a vertex is surrounded by the vertices with degree $\langle k_{nn} \rangle$ as illustrated in Fig.2. First we need to examine what happens on the vertex with $\langle k_{nn} \rangle$. In mean field picture, as a vertex is surrounded by vertices $\langle k_{nn} \rangle$, so the vertex $\langle k_{nn} \rangle$ is surrounded by the vertices $\langle k_{nn} \rangle$, which is illustrated in Fig.2 and called $\langle k_{nn} \rangle$ network hereafter. Next, we study the fixation problem on the $\langle k_{nn} \rangle$ network. The point is that this network is regular network with degree $\langle k_{nn} \rangle$. We are able to use the results of previous studies[11,12], that is, on regular network with degree k the condition of favoring cooperation is $b/c > k$. Thus, on $\langle k_{nn} \rangle$ network the condition is $b/c > \langle k_{nn} \rangle$. It is necessary for the whole network to be fixed that at least $\langle k_{nn} \rangle$ network is fixed. Unless $\langle k_{nn} \rangle$ is fixed, the whole network is not fixed either. Thus that is the necessary condition. However, we will see that it is not only necessary condition but also sufficient condition in mean field picture. What happens if $\langle k_{nn} \rangle$ network is fixed? Remind the rule of the game, for each time step one vertex is randomly chosen to die and the adjacent vertices compete for the empty vertex proportional to their fitness. In mean field picture, any chosen vertex is surrounded by the vertices with degree $\langle k_{nn} \rangle$. Because we assume that $\langle k_{nn} \rangle$ is already fixed, any vertex surrounding randomly chosen vertex is a cooperator. Since the empty site will be taken by the adjacent players and all of them are cooperators yet, the agent on the site has no choice but to be a cooperator. As we have seen, we only have to study the fixation problem on the $\langle k_{nn} \rangle$ network in our mean field picture. The condition of favoring cooperation on the $\langle k_{nn} \rangle$ network is $b/c > \langle k_{nn} \rangle$. As we have discussed, this is also the condition for the whole network. On regular networks $\langle k_{nn} \rangle$ is always equal to k, thus the rule becomes $b/c > k$.

Because we have the condition $b/c > \langle k_{nn} \rangle$, we can compare how three kinds of representative networks: regular networks, random networks[26], and scale free networks favor cooperation most. Scale free network is the network that has degree distribution $P(k) \sim k^{-gamma}$. We need to fix $\langle k \rangle$ across the networks. Even though $\langle k \rangle$ are the same, $\langle k_{nn} \rangle$ can be different. This fact enables us to compare across different kinds of network. Among three networks, regular network favors cooperation most and scale free network opposes most. On ideal scale free network which has infinite number of vertices with $2 < \gamma \leq 3$, many real scale free network falls this parameter range, cooperation is unfeasible because $\langle k_{nn} \rangle \to \infty$.

We check two points whether the condition $b/c > \langle k_{nn} \rangle$ holds well and which condition $b/c > \langle k \rangle$ or $b/c > \langle k_{nn} \rangle$ is better by numerical simulation. Seen from Fig.4 the condition $b/c > \langle k_{nn} \rangle$ holds well and is better than $b/c > \langle k \rangle$ which has been believed before. However, $\langle k_{nn} \rangle$ does not become approximate threshold for networks with too large deviation of degree distribution like scale free network Fig.4 (e). Of course the rule, $b/c > \langle k \rangle$, does not hold well for such a network either. Since it is derived by mean field approximation, it is not hard to understand that in such a case it does not hold well. We need to make another step forward to understanding the condition when the deviation of degree distribution is too large. Nevertheless, for all the networks in Fig.4 if $b/c > \langle k_{nn} \rangle$ the networks favor cooperation and this condition is sufficient. On the other hand, $b/c > \langle k \rangle$ is not a sufficient condition. There are regions where $b/c > \langle k \rangle$ satisfies but networks oppose cooperation.

Here is the conclusion. We derive the condition of favoring cooperation on networks. Although it has been believed that $b/c > \langle k \rangle$ is the condition, we show that $b/c > \langle k_{nn} \rangle$ is the one. We also show that among three kinds of network: regular network, random network and scale free network, regular network favors cooperation most and scale free network least. On scale free network with infinite number of vertices and $2 < \gamma \leq 3$ cooperation is unfeasible.

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1. Hofbauer, J. & Sigmund, K. Evolutionary game dynamics. *Bulletin-American Mathematical Society* **40**, 479–520 (2003).

2. Nowak, M. & Sigmund, K. Evolutionary dynamics of biological games **303**, 793–799 (2004).

3. Santos, F. & Pacheco, J. Scale-free networks provide a unifying framework for the emergence of cooperation. *Nature (London) Phys Rev Lett* **95**, 098104 (1992).

4. Nowak, M. A. & May, R. M. Evolutionary games and spatial chaos. *Nature* **359**, 826–829 (1992). 10.1038/359826a0.

5. Killingback, T. & Doebeli, M. Spatial evolutionary game theory: Hawks and doves revisited. *Proc. R. Soc. Lond. B* **263**, 1135–1144 (1996).

6. Nakamaru, M., Matsuda, H. & Iwasa, Y. The evolution of cooperation in a lattice-structured population. *J. Theor. Biol.* **184**, 65–81 (1997). 10.1006/jtbi.1996.0243.

7. van Baalen, M. & Rand, D. A. The unit of selection in viscous populations and the evolution of altruism. *J. Theor. Biol.* **193**, 631–648 (1998). 10.1006/jtbi.1998.0730.

8. Mitteldorf, J. & Wilson, D. S. Population viscosity and the evolution of altruism. *J. Theor. Biol.* **204**, 481–496 (2000). 10.1006/jtbi.2000.2007.

9. Hauert, C. & Doebeli, M. Spatial structure often inhibits the evolution of cooperation in the snowdrift game. *Nature* **428**, 643–646 (2004). 10.1038/nature02360.

10. Ifti, M., Killingback, T. & Doebeli, M. Effects of neighbourhood size and connectivity on the spatial continuous prisoner's dilemma. *J. Theor. Biol.* **231**, 97–106 (2004). 10.1016/j.jtbi.2004.06.003.

11. Ohtsuki, H., Hauert, C., Lieberman, E. & Nowak, M. A. A simple rule for the evolution of cooperation on graphs and social networks. *Nature* **441**, 502–505 (2006).

12. Taylor, P. D., Day, T. & Wild, G. Evolution of cooperation in a finite homogeneous graph. *Nature* **447**, 469–472 (2007). 10.1038/nature05784.

13. Lieberman, E., Hauert, C. & Nowak, M. A. Evolutionary dynamics on graphs. *Nature* **433**, 312–316 (2005). 10.1038/nature03204.

14. Szabo, G. & Vukov, J. Cooperation for volunteering and partially random partnership. *Phys. Rev. E* **69**, 036107 (2004). 10.1103/PhysRevE.69.036107.

15. Wilson, D. S., Pollock, G. B. & Dugatkin, L. A. Can altruism evolve in a purely viscous population? *Evol. Ecol.* **6**, 331–341 (1992).

16. Whitlock, M. Fixation probability and time in subdivided populations. *Genetics* **164**, 767–779 (2003).

17. Newman, M. E. J. The structure and function of complex networks. *SIAM Review* **45** (2003).

18. Dorogovtsev, S. N., Goltsev, A. V. & Mendes, J. F. F. Critical phenomena in complex networks. *Reviews of Modern Physics* **80**, 1275 (2008).

19. Konno, T. Network structure of japanese firms. hierarchy and degree correlation: Analysis from 800,000 firms. *Economics: The Open-Access, Open-Assessment E-Journal* **2** (2008).

20. Nowak, M. Five rules for the evolution of cooperation. *Science* **314**, 1560–1563 (2006).

21. Taylor, P. D. & Jonker, L. Evolutionary stable strategies and game dynamics. *Math. Biosci.* **40**, 145–156 (1978). 10.1016/0025-5564(78)90077-9.

22. Hofbauer, J. & Sigmund, K. Evolutionary games and population dynamics (1998).

23. Nowak, M. A., Sasaki, A., Taylor, C. & Fudenberg, D. Emergence of cooperation and evolutionary stability in finite populations. *Nature* **428**, 646–650 (2004). 10.1038/nature02414.

24. Pastor-Satorras, R. & Vespignani, A. Epidemic spreading in scale-free networks. *Physical Review Letters* **86**, 3200 (2001).

25. Albert, R., Jeong, H. & Barabasi, A. Attack and error tolerance of complex networks. *Nature* **406**, 378–382 (2000).

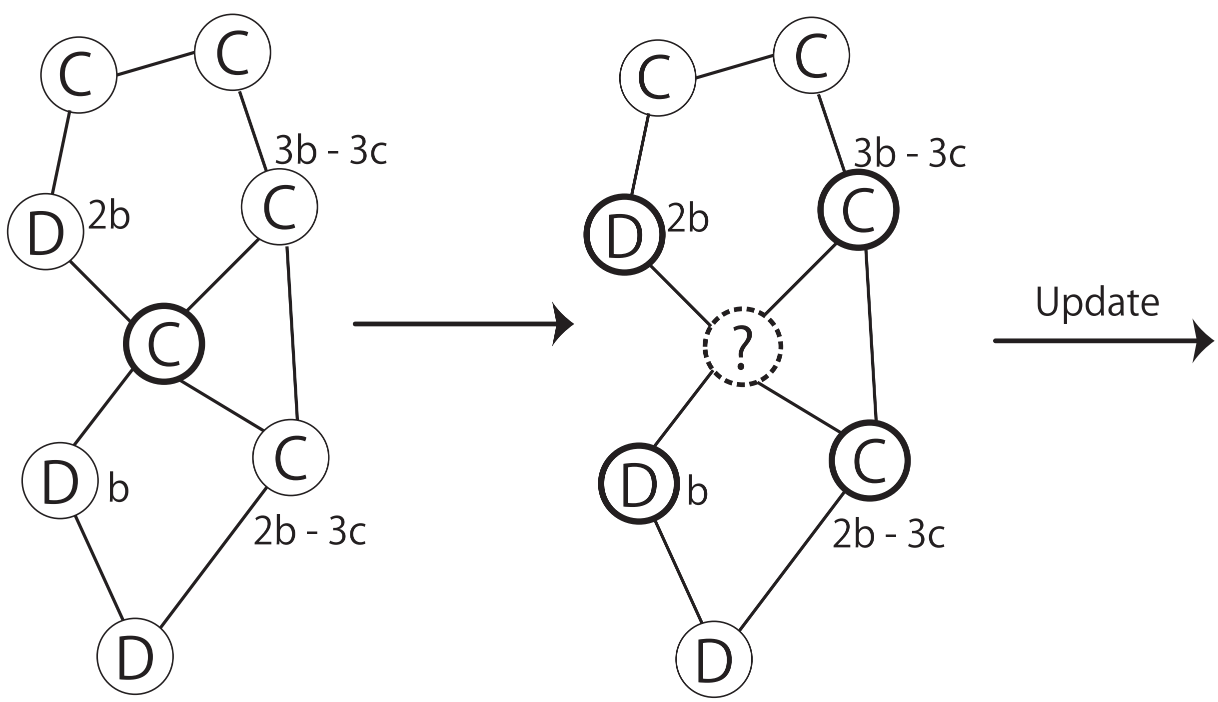
26. P.Erdos & A.Renyi. On random graphs. *Publicationes Mathematicae* **6**, 290–297 (1959).

Figure 1 The rule of the game. All the vertices are occupied by individuals. An individual derives payoffs, P, from the game. The fitness is $1 - w + wP$, where $w \ll 1$ means weak selection. $w = 1$ means strong selection. For each time step, randomly chosen individual dies, then the empty vertex will be occupied by the neighbors in proportional to their fitness. In the figure, the fitness of cooperators is $F_C = 2 - 2w + w(5b - 6c)$. The fitness of defectors is $F_D = 2 - 2w + 3wb$. The vertex will be occupied by a cooperator with probability $F_C/(F_C + F_D)$.

Figure 2 Mean Field. (a) In mean field approximation, every vertex adjacent to any vertex with arbitrary degree, k, is replaced by the vertex with degree $\langle k_{nn} \rangle$. (b) The vertex with $\langle k_{nn} \rangle$ is also surrounded by the vertices with $\langle k_{nn} \rangle$.

Figure 3 Intuition for the rule. Suppose that all the vertices with $\langle k_{nn} \rangle$ are occupied by cooperators as illustrated in (b). Because every vertex is surrounded by the vertices with degree $\langle k_{nn} \rangle$ in mean field picture and they are already occupied by cooperators as supposed, any vertex with arbitrary degree, k, is surrounded by cooperators. Remind the rule of the game that an empty site will be occupied by the neighbors. Now, all the neighbors of any vertex are cooperators in mean field picture, subsequently overall network will be occupied by cooperators.

Figure 4 Numerical Simulations. N stands for the size of networks. y-axis means ratio defined by the fixation probability of a neutral mutant, $1/N$, divided by the fixation probability of cooperator. Dashed lines are ratio=1. If ratio is less than 1, the network favors cooperation. For each simulation, the value of b/c is $(\langle k_{nn}\rangle - \langle k\rangle)t + \langle k\rangle$. t is the control parameter. $t = 0$ corresponds to $b/c = \langle k\rangle$, while $t = 1$ corresponds to $\langle k_{nn}\rangle$. x-axis means t. For every simulation, after each $2N$ time steps we build new networks because the fixation probability may be dependent on specific realization. The games are under weak selection, $w = 0.01$. The values of $\langle k\rangle$ and $\langle k_{nn}\rangle$ are merely examples, they vary across realizations. (a) This network is the mixture of vertices with degree 5, 6, 7, 8, 9, and 10. The number of five kinds of vertices are the same. We iterated 400000 steps for each. $\langle k\rangle = 7.5, \langle k_{nn}\rangle = 7.9$ (b)-(d) Random networks with $\langle k\rangle = 10, 12, 14$. N=600, 600, 700. Iteration is 72000, 72000, and 84000. $\langle k_{nn}\rangle = \langle k\rangle + 1$ (e) Scale free network with $\gamma = 2.2$ and $N = 500$. $\langle k\rangle = 3.7, \langle k_{nn}\rangle = 6.6$ From $t = 1$ to $t = 0$, iterations are 25000, 25000, 25000, 50000, 100000, and 150000.



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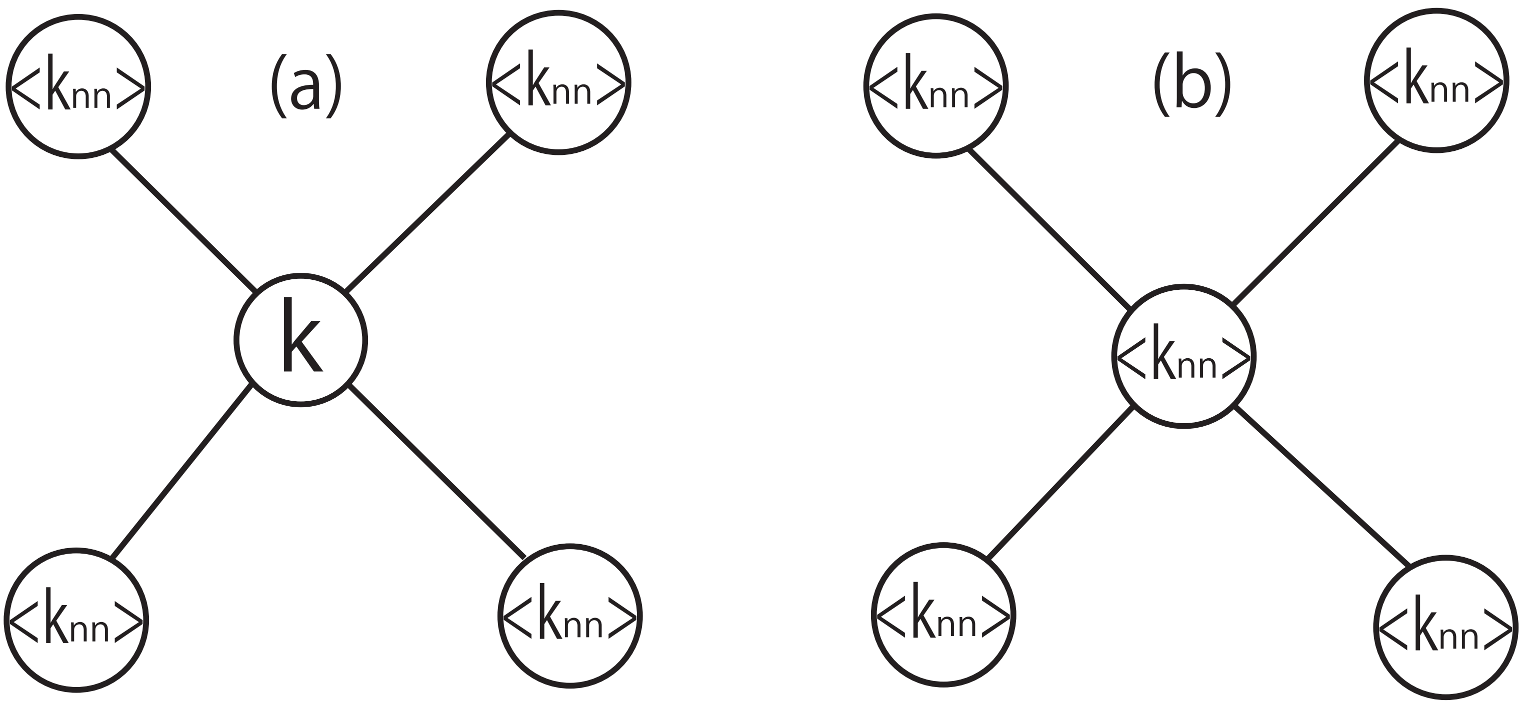
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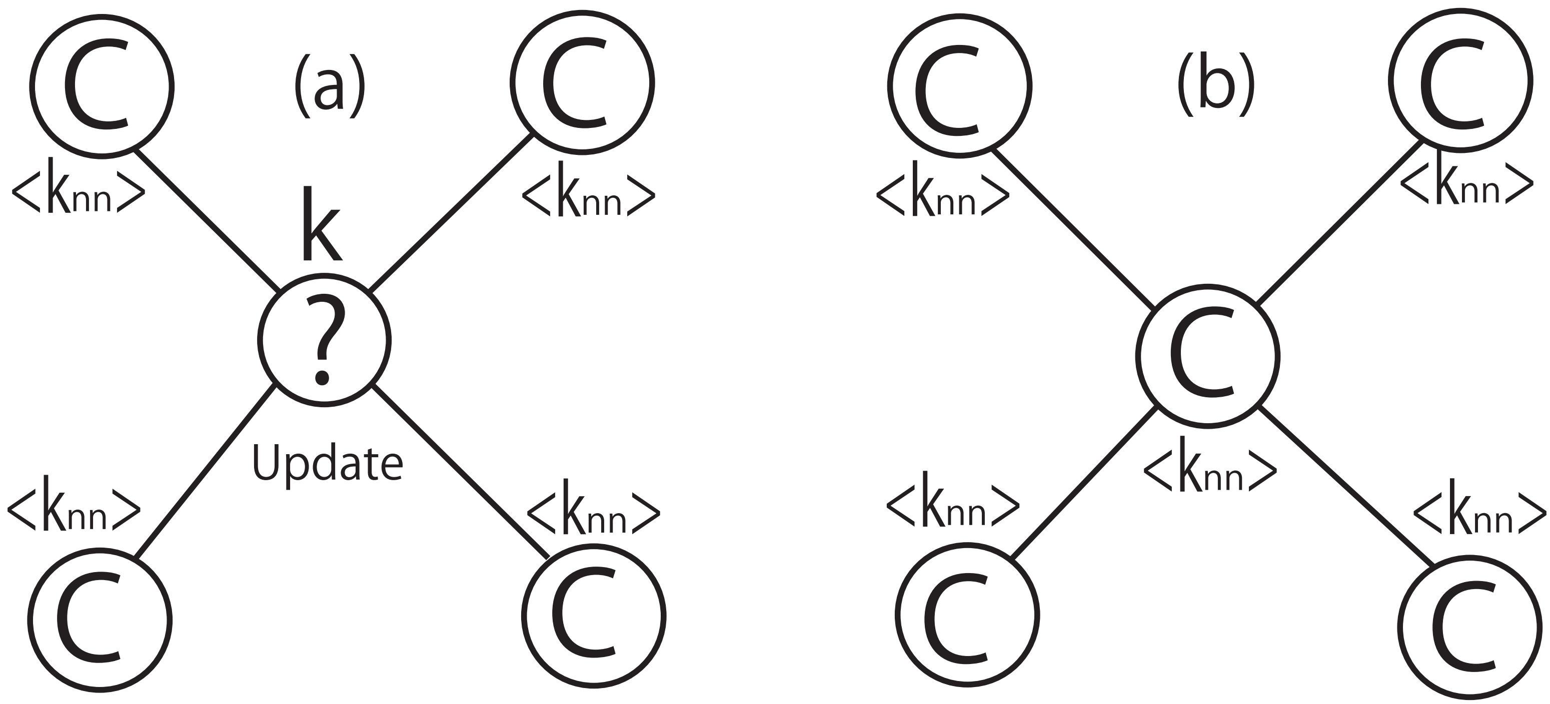
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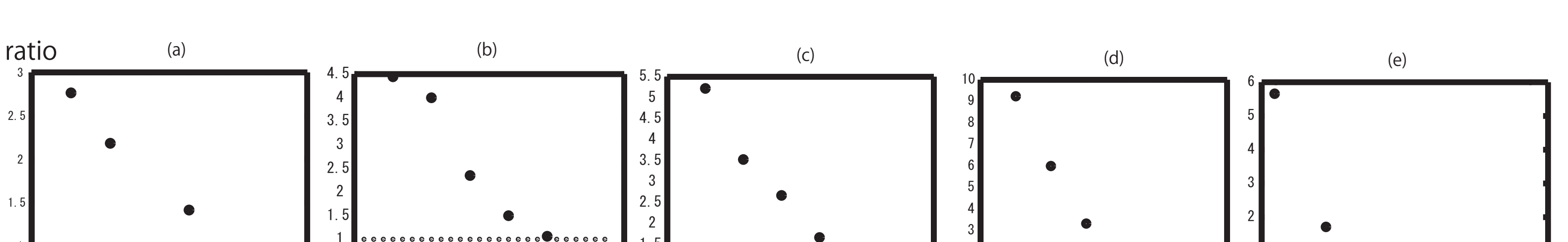
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