Can chance be less than zero?

An intriguing review of physical problems yielding negative probabilities may fail to demonstrate their reality, but should not be overlooked on that account.

By simple definition, as everybody knows, the values assigned to quantities representing the probability of events must be positive real numbers. How could it be otherwise? The chance that a flipped coin will come down showing heads, not tails, is roughly 0.5, the probability of the only other possible outcome is roughly the same, and the two numbers added together make exactly 1.0, which is the probability that there will be an outcome of one kind or the other. In such a context, the notion that probabilities might have negative values is entirely devoid of meaning. That is what everybody knows.

None of this has deterred Dr W. Muckenheim of Göttingen from compiling a substantial and fascinating review of negative probability, called "extended probability" no doubt so as to avoid the charge of sensation-mongering (*Physics Reports* **166,** 337; 1986). Part of Muckenheim's case is that if it has been useful to extend the set of natural numbers to include negative and irrational and imaginary numbers, to allow systems to have negative energy (as in some solutions of Dirac's equations) and even negative temperatures (as in laser and maser materials with artificial populations of excited states), there can be no good reason why probabilities should be sacrosanctly positive. His more tangible starting point is a puzzle which has been about in quantum mechanics since it was first identified by Eugene Wigner in 1932.

Muckenheim comes to no particular conclusion about the meaning of negative probability. Properly, he notes that the question is almost certainly linked with other current imponderables, such as the question whether quantum mechanics requires a revision of classical causality or even of the principles of logic (as first suggested by von Neumann); tinkering with the meaning of probability may be a simpler task.

But instead of a conclusion, Muckenheim has assembled the opinions of several of those who have contributed to the field, including refreshingly those of E.T. Jaynes of Washington University, St Louis, and M.S. Bartlett, long-since retired from the University of Manchester but plainly doing more than merely growing roses at his private address in Devon.

Wigner's conundrum arises from his attempt (with Szilard) to calculate for a general quantum system the joint probability distribution of the coordinates which describe it and of the conjugate momenta. Although the uncertainty principle disallows the simultaneous exact measurement of these quantities, it is clear that there must be conjugately linked probability distribution functions for position and momentum.

For a system with only one degree of freedom with a position coordinate x and momentum p, for example, is is a simple matter to calculate the properties of the momentum distribution function for any stationary state of the system for which the wavefunction is known, say as $u(x)$. For then the successive moments of the momentum distribution function can be calculated directly from the wavefunction as the integral over the range of x of the function $u^*(x)\mathbf{p}u(x)$ where * denotes the complex conjugate of the wavefunction and $\mathbf{p}$ is the momentum operator $(\hbar/i)$ (d/dx).

So why not calculate the distribution function that gives the joint probability that x is within a certain interval and p the scalar value of the momentum within another? This is what Wigner did in 1932. The result was a distribution function (for one degree of freedom) $P(x,p)$ which, apart from a factor $1/(pi.\mathrm{h})$, is the integral over the dummy variable y of the quantity $u(x+y).u(x-y).\exp(2ipy/\hbar)$. The result, which is obviously generalized to systems with more than one degree of freedom, is a distribution function with some of the right properties, but which also has the disconcerting property of being negative over some parts of the range of x and p. Muckenheim gives an interesting account of the circumstances in which the Wigner function and its many generalizations misbehave in this way.

This is not the only connection in which negative probabilities are known to occur. Relativistic generalizations of classical quantum mechanics habitually yield negative energies and, with them, functions which it is tempting to interpret as probability distributions that are capable of being negative, so much so that Dirac is quoted as believing them to be inseparable. They also crop up in field theories, as for example is chemes for accounting for the interaction between real particles by the emission and absorption of "virtual" field particles. Similar difficulties arise in the discussion of the Einstein–Rosen–Podolski paradox, the *Gedankenexperiment* designed in 1935 as a challenge to Bohr's dictum that physical variables that do not commute with each other cannot in any circumstances be measured simultaneously, and from which the more recent discussions of Bell's inequality and Aspect's experimental tests of quantum correlations have flowed. Muckenheim proves, by demonstration, his belief that negative probabilities signal connections with the important problems of physics.

But what do negative probabilities mean? Bartlett (who wrote a paper on negative probability in 1945) provides one simple way of fixing ideas: a convenient way of handling the probability that a biased coin will fall on one side or the other is to represent the chance that it will turn up heads as $0.50 + p$, in which case the extra variable is plainly capable of being negative as well as positive. Even so, Bartlett was able to show (in 1945) that quantities representing probabilities which can take negative values can be manipulated by most of the usual rules of the probability calculus. His view then has not changed much over the years; most of the difficulties with negative probabilities appear to arise from people's determination to follow Max Born in the interpretation of the square of the wave function as a probability density, for which he sees no physical justification.

Running through this commentary on negative probability is the opinion that, in practice, sensible people will ensure that they do not interpret probability distributions physically unless they have somehow manipulated them so as to be everywhere positive.

Other ways of squaring this circle are more ingenious. Dewdney, Holland, Kiprianidis and Vigier from the Poincaré Institute in Paris suggest that it might be worthwhile to consider negative probability distributions as the difference of two separate and presumably positive densities which may, for example, represent the distributions of particles and antiparticles respectively.

Gallantly for one who clearly wishes that somebody would give meaning to negative probability, Muckenheim gives almost the last word to Jaynes who, while sharing the general doubt about the validity of Born's interpretation of the square of the wave equation as a probability density, makes the simple point that the onus of proof rests on those who talk about negative probability. That, unfortunately perhaps, will be the general view.

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