Observation and things observed

A persuasive account has at last been given of the probability spread of quantities that cannot simultaneously be measured: measuring devices enter equally with the systems measured.

EVERYBODY knows that it is not possible simultaneously to measure accurately the position and the momentum of a particle. So much has gone without saying for the past half-century, and the same applies to the members of any other pair of canonically conjugate dynamical variables, say energy and time. More generally, the minimum uncertainties of two simultaneous measurements of, say, position and momentum are inversely related to each other in the sense that the product of the two uncertainties is not less than some constant which turns out to be Planck's constant divided by 2π, called $\hbar$ for short. This is how Heisenberg's Uncertainty principle has instructed us since 1925.

Nothing in what follows is to be read as a contradiction of Heisenberg's principle but only as a gloss on it prompted by an intriguing discussion of the problem of simultaneous measurement by K. Wodkiewicz of the University of Rochester (*Phys. Rev. Lett.* **52**, 1064; 1984). One way of putting his question is to ask what can be said about the statistical distribution of the momentum of some particle, say an electron, whose motion is sufficiently well understood for it to be represented by a wave function which is a solution of Schrödinger's equation. The standard solution for the simplest case of all, a particle moving freely through a force-free region, is elementary — the wave function is a plane wave, and the implication is that the particle is equally likely to be found anywhere in the infinite accessible region. The momentum distribution is equally simple — it consists simply of the momentum corresponding numerically to the wave vector of the plane wave — a single point in momentum space.

More complicated dynamical systems naturally lead to a more complicated dual relationship between the probability distribution of a position coordinate and its conjugate momentum. But if a wave function (in terms of the position coordinate) can be found for the stationary state of the system, it is always possible to calculate the expected value of the momentum, and the successive moments about this mean, by the simple rules of quantum mechanics or wave mechanics. But why not go the whole hog, and from the outset calculate the probability distribution in some two-dimensional space (phase space) whose dimensions represent position (q) and momentum (p)?

Wodkiewicz points out that this question was first tackled in 1932 by Eugene Wigner, but a footnote to that paper (*Phys. Rev.* **40**, 749; 1932) remarks cryptically that Wigner's solution of the problem "was found by L. Szilard and the present author some years ago for another purpose". Wigner's limited objective in 1932 was to find some way of handling functions representing dynamical systems in which each of a pair of conjugate variables appears, in particular the expression for the total energy of the system consisting of the sum of its kinetic energy (a function of p) and potential energy (a function of q). Wigner's goal was to correct classical Boltzmann statistics for the effects of quantum mechanics that become apparent at low temperatures. The byproduct of his work was a function of both q and p, called $W(p,q)$, with most of the properties that would be expected of a probability distribution in phase space.

Wigner's 1932 result defines the combined distribution of a pair of conjugate variables in terms of the wave function with respect to only one of them, and seems to have stood the test of time in spite of its deficiencies. If, for example $f(q)$ is a wave function for a particle moving in one direction, and thus a solution of Schrödinger's equation, the probability distribution of q if given by $f^*(q)f(q)$ (where the asterisk denotes the complex conjugate and where it is assumed that the wave function has been multiplied by some number that makes the total probability unity). Wigner's dual function $W(p,q)$ is $(\pi\hbar)^{-1}\int f^*(q+t)\ f(q-t)\ \exp(2ipt/\hbar)\ .\ dt$ and was shown to have many of the properties required of a joint probability distribution. Integrating over one variable would, for example, give the probability distribution for the other.

The only other drawback, which Wigner pointed out, is that the function W can sometimes be negative, and so cannot be a true probability. Even so, nearly half a century later Wigner (with R. F. O'Connell) argued that W is the only function that will satisfy what reasonable people would require of a joint distribution (*Phys. Lett.* **83A**, 145; 1981).

Not so, says Wodkiewicz, who takes an admirably positivist point of view. There is no point in talking about phase-space until procedures have been specified for measuring its coordinates, position q and momentum p. He suggests how the job can be done using a hypothetical pulsed laser whose radiation can interact with a particle by means of its electrical field.

The consequence of the interaction is that a detected particle is scattered so that the form of its wave function is such that, at a great distance from the point of interaction, information can be gleaned both about position and momentum, but only as allowed by the uncertainty principle. This is the basis of what Wodkiewicz calls his operational definition of a phase-space probability distribution.

Everybody, but not least Professor Eugene Wigner, will be anxious to know how the result resembles the old $W(p,q)$. Put briefly, there are on this occasion two wave functions to be taken account of, one describing the state of the laser beam with which the detected particle interacts (f, say) and the other describing the state of the particle after the interaction (here called g), each of which can be used separately to define a Wigner-like phase-space distribution which may be called $W_f(p,q)$ and $W_g(p,q)$ respectively.

Wodkiewicz's result is geometrically very simple (for one space coordinate). The value of the probability distribution in phase-space at the point (p,q) is obtained by displacing the origin of one Wigner function (a pattern in two dimensions) to that point, multiplying by the other undisplaced Wigner function and then integrating over both variables. In simple language, the result is the overlap between two Wigner functions, one for the measuring system and one for the particle. Conveniently, the result cannot be negative.

This outcome is satisfactory for several reasons, but not least because it puts the observing system and the thing observed on an equal footing. This is precisely what the Copenhagen school would have asked for. Second, as Wodkiewicz points out, the calculation of phase-space probability should survive even if the Schrödinger equation were non-linear (which is one way in which gravitational forces might be taken account of). It goes without saying that Wigner's result can be used more confidently now that the reasons why it is incomplete can be understood. But with all that said, is it not remarkable that after half a century of careful examination of the meaning of quantum mechanics, during which the notion that the measuring instrument must be counted part of the system measured has been repeatedly raised, it is only now that the deficiencies of Wigner's calculation should have been explained?

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