RAMANUJAN

By Prof. L. J. Mordell, F.R.S.

PROF. G. H. HARDY who, some thirteen years ago, supervised the editing of Ramanujan's collected papers, has now produced a new volume dealing with Ramanujan.* It is a series of essays rather than a systematic account of his work. However, it includes most of his more important discoveries, and also much recent work associated with the results found by Ramanujan. There are twelve essays, and it will make my subsequent remarks more intelligible if I give their titles in full: (1) The Indian mathematician Ramanujan; (2) Ramanujan and the theory of prime numbers; (3) Round numbers; (4) Some more problems of the analytic theory of numbers; (5) A lattice point problem; (6) Ramanujan's work on partitions; (7) Hypergeometric series; (8) Asymptotic theory of partitions; (9) The representation of numbers as sums of squares; (10) Ramanujan's function $\tau(n)$; (11) Definite integrals; (12) Elliptic and modular functions.

I may as well say at once that the book is beautifully written in an informal style, and reads as smoothly and fascinatingly as a delightful novel. The simplicity and clearness of the style, and the care in setting out the formulæ, are so marked everywhere as to make it plain that the book is certainly a labour of love on which its author has spared no effort. Proofs are given with a minimum of unpleasant detail, with the right emphasis on the salient features and with useful and interesting comments and notes. The book is a model of good exposition. It is a handsome quarto volume printed in such a way as to make it a pleasure to read and a credit to the printers.

The main facts of Ramanujan's career are well known, and the lives of few other mathematicians for some generations past have been so full of human interest. The story of his life is that of the rise of an obscure Indian in the face of the greatest difficulties to the position of the most famous mathematician that India has ever produced and of his early death just after he had won the most coveted distinctions. The publicity given to him suggests that the world likes to hear about mathematicians; and in fact Lord Riddell, the journalist peer, who heard of Ramanujan from Mr. Montagu, at one time Secretary for India, was so interested that he sent for a copy of Ramanujan's collected works and wrote a review, confining himself, of course, to an account of Ramanujan's life.

* Ramanujan: Twelve Lectures on Subjects suggested by his Life and Work. By Prof. G. H. Hardy. Pp. vii+236. (Cambridge: At the University Press, 1940.) 25s. net.

Ramanujan was born in 1887 near Kumbakonam, a fair-sized town about 160 miles from Madras. Like many other mathematicians, he came of poor and humble folk. His schooling was normal, though at an early age he was recognized as a boy with exceptional abilities. He went to the local high school from the age of seven to sixteen, and in 1904 entered the Government college at Kumbakonam, where he won a scholarship. In no other science as in mathematics is it so easy for a student's interest to be aroused at an early age. The simplest properties of numbers which occur in arithmetic, or the simplest properties of geometrical figures, have often sufficed. Apparently Ramanujan's interest displayed itself when he studied trigonometry and found for himself results given in Part 2 of Loney's "Trigonometry." Then, at the age of sixteen, he borrowed an old book by Carr, "A Synopsis of Elementary Results in Pure and Applied Mathematics". This covers roughly the subjects of Schedule A of the present Mathematical Tripos, and contains the enunciation of some six thousand theorems with proofs that are often little more than cross-references. Carr has sections on algebra, trigonometry, calculus and analytical geometry, and emphasizes in particular the formal side of the integral calculus. It was this book that really awakened Ramanujan's genius. It secured such a grip on him that he is said to have spent all his time at college—including lecture periods in other subjects—upon his mathematics. Many years ago Huxley (and probably many others before him) said that one of the most important objects of any educational system should be to catch the small percentage of the population with some special aptitude, to turn them to account for the good of society, to see that they are not starved by poverty, and that they are put in the positions in which they can do the work for which they are specially fitted. If Ramanujan had lived in Britain he could have started early specialization in preparation for a scholarship examination at one of the universities, with what wonderful results no one can imagine. Unfortunately, the educational system at Kumbakonam was not elastic enough to deal with persons like him. As a result, his neglect of college work other than mathematics led to disaster. He lost his scholarship, left college, ran away from home, came back, returned to college, but did not make up for his absence. He then entered Pachaiyappa's College, Madras, in 1906, but, falling ill, returned to Kumbakonam. He appeared as a private student for the F.A. examination in December 1907 and failed. In England he would probably then have been in the middle or near the end of his college career, with possibly the Tripos before him, and also an immediately successful career.

But from 1907 until 1912, Ramanujan was adrift in the world without any definite occupation except his mathematics, which must have absorbed most of his energy. The results of this are embodied in some now famous note-books. In 1909 he married and so, as remarks his biographer, he had to find some regular employment. He had great difficulty in finding any because of his unfortunate college career. In 1910 he began to find more influential friends, who tried in vain to find a tolerable position for him. But in 1912, at about the age of twenty-five, he became a clerk in the office of the Port Trust of Madras at a salary of about £30 per annum. Not much has been published about his life during these critical years 1907–1912. His first substantial paper had been published in 1911 in the *Journal of the Indian Mathematical Society*. In 1912, he began to secure some recognition in India. It is really an easy matter for anyone who has done brilliant mathematical work to bring himself to the attention of the mathematical world, no matter how obscure or unknown he is or how insignificant a position he occupies. All he need do is to send an account of his results to a leading authority. One can recall the classic instances of Jacobi's letters to Legendre announcing his discovery of the elliptic functions, and of Hermite's letters to Jacobi containing his new discoveries in number-theory.

Ramanujan wrote to Hardy in January 1913, sending him the enunciation of a great many results he had found, many of them strikingly original and thoroughly intriguing, others well known, and some false and yet not without considerable interest and significance. Events had begun by this time to take a more favourable turn in India. The University of Madras gave him a scholarship of some £60 per annum, adequate for a married Indian, and he was also sounded about a trip to England, which he declined. In 1914, however, he was prevailed upon to go to Cambridge with help from the University of Madras and from Trinity College. There he had three years of uninterrupted activity and continuous contact with Hardy, the results of which are visible in his "Collected Papers." He fell ill in 1917, and never fully recovered, dying in 1918 shortly after his election to the Royal Society and to a fellowship at Trinity College. He was the first Indian to have been awarded either honour.

It may well be asked at once what kind of mathematics could be done by a person with Ramanujan's training, especially before he came to England, and what characteristics his talents display. Mathematical research is possible in many directions, and two extreme ones suggest themselves. In one, for example, modern algebra, it is necessary to master a technique requiring considerable preparation and study, and research is practically impossible without this. In the other comparatively little pre-knowledge is required; only native wit and exceptional ability and intuition. This applies especially to identities involving infinite series, products, continued fractions and integrals. Probably many of the results found will not be new.

Thus the simply periodic functions, for example, those for which $f(x+1)=f(x)$, and in particular the elementary trigonometric functions, lead to the well-known series and products associated with the sine function, as well as to the familiar Bernoulli numbers in the expansion of $1/(e^x-1)$ in ascending powers of x. Then the hypergeometric series

$$\sum_0^\infty \frac{a.\ a+1.\ \ldots\ a+n-1.\ b.\ b+1.\ \ldots\ b+n-1}{n!\ c.\ c+1.\ \ldots\ c+n-1}\,x^n,$$

which includes most of the series occurring in elementary mathematics, lends itself to all sorts of extensions and generalizations. This series is also closely associated with the gamma function $\Gamma(x)$. This function, the characteristic property of which is $\Gamma(x+1) = x\,\Gamma(x)$ and which is a generalization of $x\,!$, is rich in applications to identities of the kind mentioned above.

There are also two classes of far more abstruse functions of the greatest importance in the mathematics of the last century. One is the elliptic or doubly periodic functions, the chief property of which may be typified by

$$f(x+1)=f(x), \qquad f(x+\omega)=f(x),$$

where ω is a complex constant. The simplest elliptic functions are those called $\wp(z)$, sn z, cn z. From the eliptic functions arise the modular functions, of which the simplest have the curious general periodicity property that if ω is a complex variable and α, β, γ, δ are any integers such that $\alpha\delta - \beta\gamma = 1$, then

$$f\left(\frac{\alpha\omega+\beta}{\gamma\omega+\delta}\right)=f(\omega).$$

The function which naturally arises in studying such functions can be written as

$$\Delta\,(\omega_1, \omega_2) = (2\pi/\omega_2)^{12}q^2 \prod_1^\infty (1-q^{2n})^{24}.$$

Here ω_1, ω_2 are two complex numbers whose ratio $\omega = \omega_1/\omega_2$ has a positive imaginary part, and $q = e^{\pi i\omega}$ so that $|q| < 1$. Then

$$\Delta\,(\omega_1, \omega_2) = \Delta\,(\alpha\omega_1+\beta\omega_2, \gamma\omega+\delta\omega_2).$$

The general theory of elliptic and modular functions has been treated in monumental works by Klein and Fricke, and by Weber, and reveals at once the most beautiful and fascinating series and products, and enables us not only to prove directly any relevant identities but also to state beforehand their nature and form. Many of these identities can be found by quite elementary means and others by the display of great ingenuity. There may then be no connexion with the general theory, and the results may appear to one unfamiliar with it as an extraordinary collection of strange, curious and isolated results, an impression sometimes heightened by the unusual form in which they can be expressed.

Finally, the theory also shows that if $f(\omega)$ is a modular function and n is any positive integer, then algebraic equations of great interest called modular equations connect $f(n\omega)$ and $f(\omega)$, and that these can be expressed in a multiplicity of ways. Further, if we put $f(n\omega) = f(\omega)$, which then involves $n\omega = (\alpha\omega + \beta)/(\gamma\omega + \delta)$, so that ω is a complex quadratic surd, the equations are solvable by radicals, as was shown by Abel about a century ago. Some of these equations can be solved very simply, while the solution of others involves a wonderful collection of surds.

It was chiefly in these subjects that Ramanujan's best work was done. They gave him ample scope for his exceptional and brilliant genius, which displayed such wonderful imagination, intuition and insight. For formal manipulation of infinite processes and an instinctive feeling for algebraical formulae, he was unrivalled since the time of Euler and Jacobi. His fertility in producing a host of strange and curious results was unbounded and ceased only with his death.

I mention first some results illustrating his inductive powers.

Suppose that q is a complex number and $|q| < 1$. Write

$$1/\prod_1^\infty (1-q^n) = 1 + \sum_1^\infty p(n)q^n.$$

Then the coefficient of q^n in the series is a very important function of n known as the partition function, of which more later. It is easily shown to be the number of solutions in positive integers $x, y, z, t, \ldots$ of

$$x + 2y + 3z + 4t + \ldots = n.$$

Though it had been known and studied since the time of Euler, very little was known of its properties; for example, it is even now not known when $p(n)$ is even or when it is odd. Ramanujan found by observation that $p(5n + 4)$ is divisible by 5, and then proved it. Though no one else had noticed this result, it is easily suggested to anyone studying the numerical values of $p(n)$. A proof is also obvious from the identity

$$p(4)+p(9)q+p(14)q^2+\ldots = 5\{(1-q^5)(1-q^{10})(1-q^{15})\ldots\}^5\{(1-q)(1-q^2)(1-q^3)\ldots\}^{-6},$$

due to Ramanujan who, however, never published a proof. Many such identities, though striking in appearance, are in fact simply particular results in the theory of the modular functions.

Another result on a rather different footing found inductively without proof by Ramanujan is

$$1+\sum_1^\infty \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)} = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\ldots}.$$

What seems remarkable in this formula is that in the product on the right-hand side, the indices of the powers of q are of the form $5n \pm 1$, while in the series on the left-hand side, 5 plays no part in the indices. Such an identity seems very difficult to discover empirically, but still it might be possible for someone without his genius. The formula, which seemed difficult to prove, was in fact originally found and proved by Rogers and then overlooked for some twenty years.

The most remarkable result Ramanujan found inductively, again without proof, is one which would have occurred to very few people indeed. Write

$$q\{(1-q)(1-q^2)(1-q^3)\ldots\}^{24} = \sum_1^\infty \tau(n)q^n,$$

where the product is associated with the important modular function $\Delta(\omega_1, \omega_2)$, so that it can be expected that the coefficients $\tau(n)$ are also important. Then Ramanujan conjectured that

$$\sum_1^\infty \tau(n)n^{-s} = \prod_p (1-\tau(p)p^{-s}+p^{11-2s})^{-1},$$

where the right-hand product is extended over all primes p. This identity implies in particular that $\tau(mn)=\tau(m)\tau(n)$, when m and n are prime to each other. The identity was afterwards proved by Mordell, but only recently Hecke found independently the formula, the proof, and important extensions. Ramanujan also conjectured that if p is a prime number, $\tau(p) \leq 2p^{\frac{11}{2}}$, whence it follows that $\tau(n)$ is of order of magnitude $n^{\frac{11}{2}+\epsilon}$, but no one has ever proved this. Recently Rankine has got so far as $n^{\frac{29}{5}}$.

The most pregnant result stated by Ramanujan is that the coefficient of x^n in $(1-2x+2x^4-\ldots)^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh \pi\sqrt{n} - \frac{1}{\pi\sqrt{n}}\sinh \pi\sqrt{n}\right).$$

This statement is false, but the formula is a genuine approximation, though not so close as Ramanujan imagined. This result, included among those sent by Ramanujan to Hardy in 1913, obviously led to the corresponding problem for the function $p(n)$ and to their joint work on partitions. This was the beginning of the method. developed by Hardy and Littlewood, for solving problems dealing with the partitions of numbers in various ways, and transformed later almost out of recognition by Vinogradoff. Thus Vinogradoff has proved that every sufficiently large odd integer is the sum of three primes. The final result of the original application of the method was to obtain a series giving exceedingly good and rapid approximations to the value of $p(n)$; for example, when $n = 200$, the error made in taking eight terms of the series gives an error of only 0·004 in the value of $p(n)$, a number of thirteen digits. Rademacher has recently replaced the asymptotic series by an equally effective convergent series.

In hypergeometric series, there is an important and fundamental formula involving many parameters due to Dougall; this was rediscovered by Ramanujan, and probably the methods he used would have led to a rigorous proof. From it he deduced a host of other results. He was in possession of yet others, suggesting that he had not revealed all the results in his possession.

Definite integrals greatly interested Ramanujan. Many of his results were found before he came to England and while he held a research scholarship at the University of Madras. It will suffice to mention the following:

$$\int_0^\infty x^{s-1}(f(0)-xf(1)+x^2f(2)-\ldots)dx = \frac{\pi}{\sin\pi s}f(-s).$$

The result is a purely formal one. It is obviously not true unless the function $f(x)$ satisfies appropriate conditions, since the formula implies the false result that $f(s)$ is identically zero when $f(0)=f(1)=f(2)=\ldots=0$. Often with results found formally without rigorous proof, it is a routine matter to obtain such a proof, but this does not apply to the present one. As shown by Hardy, the proof involves ideas and methods of which Ramanujan knew nothing in 1914 and which he had scarcely absorbed before his death. The formula is a really interesting one suggesting many other results, some of which can be proved in other ways. I wish that Prof. Hardy had given an account of other definite integrals evaluated by Ramanujan, for example, those associated with Gauss's sums. For Ramanujan was the first to evaluate, in a characteristically original way, definite integrals such as $\int_0^\infty \frac{\cos tx}{\cosh x} \cos mx^2\, dx$, which had not been done by writers such as Kronecker and Hardy himself who had both studied related integrals.

This seems to be the place to speak of Ramanujan's characteristic trait, which made him in his earlier days before his stay at Cambridge almost unique among mathematicians. A mathematical theorem is invariably associated with proof, and no mathematician would be satisfied with a result unless he had a proof. It may happen that he is led to believe in the truth of results which he cannot prove, but this in no way diminishes his desire to find proofs, which are ever his goal. But as Littlewood says: "Ramanujan had no clear cut conception of proof; if a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further." This is all the more surprising as the idea of proof is so fundamental even in elementary mathematics. But even in his very first long paper already referred to, which deals with Bernoulli's numbers, after writing down the values of twenty of them, he states that it will be observed, *inter alia*, that the numerator of $B_{2n}/2n$ in its lowest terms is a prime number. He takes the numerical evidence as sufficient, and there is no trace of any suggestion that there is need of other proof of these results, which as it happens, are well known. Proofs of many of the results stated in his notebooks and letters were given afterwards by Hardy and G. N. Watson.

Ramanujan was not a well-read mathematician. In India, he apparently did not avail himself of books that were accessible to him. The only reference I find to books influencing his early work, in addition to those of Loney and Carr already mentioned, is to Edwards's "Differential Calculus" and Hardy's tract on "Orders of Infinity". There were available at Madras several books a study of which might have had a decisive influence on his work, for example, Bromwich's "Infinite Series" and Whittaker's "Modern Analysis", one of which Hardy thinks he may have seen, and also Matthews' "Theory of Numbers". He would have realized that some of the ideas expressed in his early letters were well known; such as the meaning of $\Gamma(n)$ when n is negative, and methods of attaching meanings to non-convergent series. He would have seen how vital it was to replace some of his naive points of view by more rigorous ones; for example, that a distinction must be made between $\sum_1^\infty f(n)$ and $\lim_{s\to 0} \sum_1^\infty f(n)n^{-s}$; and he would not have been so easily led astray by false analogy, as in his work on primes mentioned later. Strangely enough, he must have studied most assiduously some book on elliptic functions, probably Greenhill's. As already remarked, this subject is particularly rich in infinite series and products of a type in which Ramanujan revelled, including applications to modular functions and singular moduli. Greenhill's is an old-fashioned book, and so Ramanujan would be unaware that many of his results, strange and fantastic as they seem to those who have not a modern knowledge of the subject, are often particular cases of a general theory with which he was unfamiliar. A comparison which suggests itself is that of finding areas and lengths of curves. Before the invention of the calculus, each result proved was no mean feat. After its invention, however, the centre of interest shifted, though one might still admire now and then ingenuity displayed in obtaining special results. Needless to say, Ramanujan showed remarkable ingenuity, and gave proofs of many of his results, but of others he could have had no rigorous proof.

Ramanujan's methods were peculiarly his own. Probably no other mathematician has relied so much upon his native wit. Most mathematicians find it a great advantage to have as extensive a knowledge as possible of lines related to their own. This increases the possibility of successful research, though sometimes there is much to be said in favour of attacking difficult problems unhampered by current ideas as to the method of approach. It is futile to wonder what Ramanujan could have done with the better tools he might have had in more favourable circumstances.

Ramanujan had very little systematic knowledge of number theory. Most of his work on this subject dealt with number-theoretic functions the values or properties of which could be investigated in a non-arithmetical spirit. Prime number theory has so developed during the last fifty years as to suggest that adequate knowledge and training are indispensable for work in this field. Ramanujan had neither, with the result that most of the results to which he was led, some by false and unproved analogies (though even thinking of some of them required a vivid imagination), were either erroneous or erroneously proved, and very little was of permanent value. This applies also to an assertion in one of his first letters to Hardy that "the number of numbers between A and x which are either squares or sums of two squares is

$$K\int_A^x \frac{dt}{\sqrt{\log t}} + \theta(x),$$

where $K = 0{\cdot}764\ldots$ and $\theta(x)$ is very small compared with the integral." He later gave a false estimate for $\theta(x)$. The problem was solved by Landau in 1908 and the solution depends upon the application of the standard methods of prime number theory.

Another result mentioned in an early letter to Hardy dealt with a lattice point problem which can be expressed as follows. To find the number of positive integer values of x, y for which $ax + by \leq n$, where a, b are given and n is large. This is an important problem afterwards considered by Hardy, Littlewood and Ostrowski, and it is surprising what interesting developments arose from so innocent looking a problem.

Much more important were Ramanujan's contributions to the question of the representation of numbers as sums of squares; that is, for given k, n to find the number $r_k(n)$ of solutions in integers $x_1, x_2, \ldots x_k$ of $x_1^2+x_2^2 \; . \; . \; . \; +x_k^2=n$. If we write $\theta(q) = \sum_{m=-\infty}^{\infty} q^{x^2}$ the question reduces to that of finding other expressions for $\theta^k(q)$ which allow of simple expansions in powers of q. Thus, as shown by Jacobi,

$$\theta^2(q) = 1+4\left(\frac{q}{1-q}-\frac{q^3}{1-q^3}+\frac{q^5}{1-q^5}-\ldots\right),$$

from which $r_2(n)$ is four times the difference between the number of divisors of n of the respective forms $4m + 1$, $4m + 3$. Expansions had been found by several writers over a period of years in the case of an even number of squares, but there had seemed to be no straightforward method for establishing the formulæ. A general identity was given by Ramanujan for $\theta^{2s}(q)$ without proof but with the characteristic remark "it can be shown". The proper approach to results of this kind is the methods of the theory of modular functions discovered by Mordell, which are also appropriate for the more difficult case when the number of squares is odd, as was discovered later by Hardy.

A fair proportion of the material which Prof. Hardy has now discussed was discovered by Ramanujan in the five years after he left school, and recorded in the notebooks he kept. Some of my previous remarks have shown how unfortunate it was that he had not at an earlier age the advantage of expert guidance and supervision, so that his talents could be properly directed. This would have spared him a great deal of time spent upon known work or upon erroneous ideas, or upon unnecessary elaboration in producing a great many detailed results where perhaps a few would have been sufficient. He would have learnt the importance of rigorous proof. There is of course always the possibility that he might have proved unresponsive to guidance; and after all there is much to be said for experience of any kind that can produce results.

Though Ramanujan had a hard and difficult struggle in his earlier days, it was not so after he wrote to Hardy. In fact, he could not have found a more appreciative, a more generous or a more influential patron, collaborator and friend. Thus it is through Hardy that Ramanujan's name is associated jointly with several beautiful theorems in which he had been anticipated, for example, in the Rogers-Ramanujan identities and in the Dougall-Ramanujan hypergeometric identity. I recall that in the first edition of one of Landau's books, Kakeya's name is attached to a theorem, but the name completely disappeared in the second edition after Landau had discovered that Kakeya had been anticipated by Eneström. Again, Hardy associates Ramanujan's name in Lecture VIII of the present volume with the asymptotic formula for the number of partitions of *n*. The proof depends upon Cauchy's integral for a function of a complex variable, a subject of which Ramanujan knew practically nothing. Of course the first suggestions as to the possibility of such results and some indication of their form were due to Ramanujan, and no doubt he was full of suggestions as to the form the final results should take. There would obviously have been no such formula without Ramanujan, and one can easily understand Hardy's gratitude for having come in contact with Ramanujan.

In attempting to assess the standing of Ramanujan as a mathematician, it is difficult not to be influenced by admiration and wonder at his success in becoming a professional mathematician in spite of the greatest difficulties. The estimate depends, as for all mathematicians, essentially upon the novelty and importance of his original work. Though some of it was wrong, and some of no permanent value; though some was over-specialization of results embodied in general theory; though some of his most interesting work was anticipated or not proved, nevertheless, there remains an impressive and formidable balance which has had great influence in shaping the direction of some of the best research since his death. To very few other mathematicians are Klein's remarks made many years ago so appropriate as to Ramanujan. "The secret of gifted productivity will always be that of finding new questions and new points of view, and without these mathematics would stagnate. In a certain sense, mathematics has been advanced most by those who are distinguished more for intuition than for rigorous methods of proof."

THE COSMICAL ABUNDANCE OF THE ELEMENTS*

By PROF. HENRY NORRIS RUSSELL

EIGHTY-EIGHT chemical elements have been isolated. Their separation by chemical means is sometimes easy, sometimes very difficult, and the best available tests differ greatly in sensitivity. Spectroscopic analysis provides a test for all constituents at once; but these tests, too, are unequally sensitive. Fortunately, the two methods supplement one another.

The composition of the earth's crust—above an arbitrary depth, such as ten miles—is well known. Oxygen is the most abundant element, whether by weight or number of atoms. Silicon is next, and then aluminium, iron, magnesium, calcium, sodium and potassium. These eight elements account for 98 per cent of the whole mass—the hydrogen in the oceans for but a quarter of the remainder.

The high mean density of the earth, and the seismological evidence for a liquid core, show that the 'crust' is not a fair sample of the whole. We may hope to do better with meteorites, taking an average of the various types (stone, iron, sulphides) in the ratio of their abundance (10 : 2 : 1, according to Goldschmidt). The result is significantly, but not greatly, different. Iron, magnesium, nickel and sulphur are more abundant; silicon, aluminium, and the alkali metals less. Just these differences are to be expected if the granitic 'crust' of the earth has segregated from a main mass similar in composition to meteorites.

Recent photographic observations of bright meteors show that their orbits were elliptical and of short period, so that they are samples of the solar system rather than the cosmos. Other selection processes may have operated. The spectra of comets suggest strongly that some of the solid bodies whence the gases escape may be composed largely of carbon compounds. Such a body accompanying a stony meteorite in its flight through the earth's atmosphere would be immediately destroyed.

Outside the earth, we must rely on the spectroscope alone. In the stars, the conspicuous differences along the spectral sequence are known to arise from differences of temperature and ionization. Miss Payne's conclusion (1925) that the general run of stars are very similar in composition has been fully confirmed.

Sixty elements have been identified in the sun; and the apparent absence of almost all the rest explained by unfavourable situations of their ultimate lines in the inaccessible ultra-violet. More than forty have been identified in Pegasi.

On good *coudé* spectrograms, equivalent widths of stellar lines may be well measured; and the data for the sun are extensive. To find from these the 'effective numbers of atoms above the photo-

* Abstract of an address delivered at the symposium on September 26, in connexion with the celebration of the fiftieth anniversary of the University of Chicago.