A Neglected Aspect of the Calculus of Variations

Variationsrechnung und partielle Differentialgleichungen erster Ordnung
Von Prof. Constantin Carathéodory. Pp. xi+407. (Leipzig und Berlin: B. G. Teubner, 1935.) 22 gold marks.

THE work of Euler and Lagrange in the middle of the eighteenth century proved that the essence of theoretical mechanics lay in the calculus of variations, which was itself something that included not only all mechanics but also much else besides. Mechanics is a class of partial differential equations; the calculus of variations corresponds to a similar but bigger class, whose essential properties differ only in detail from those of the former class. Jacobi in particular stressed, in 1836, the intimate relation that subsists between systems of partial differential equations of the first order and equations arising out of the calculus of variations.

There for the next forty years the matter rested; no doubt for lack of some of those essential principles which were in that period changing from fluidity to solidity, there is small evidence of development in the direction to which Jacobi's researches seemed to point. Then in 1879 appeared the work of Weierstrass, which gave the calculus of variations its independence from its first parent, mechanics, an independence which it maintained only until it was adopted by the new and comprehensive calculus of functionals.

From the time of Weierstrass the evolution of the general theory of the calculus of variations proceeded in directions which appeared to loosen the primal association with partial differential equations, an association which, though noticed in a casual manner from time to time, all but disappeared from sight. Thus those landmarks of the period, the textbooks of Bolza (1909) and Hadamard (1910), tend away from, rather than towards, the Jacobian point of view (though Hadamard clearly realised that there was still much to be revealed), whereas the more recent and more highly specialised books—that of Morse (1934) may be quoted as a distinguished example —pass still further away.

In his preface to the book now before us, Carathéodory tells how he has been endeavouring for many years to pick up the threads of the connexion between the calculus of variations and partial differential equations left by Jacobi and his pupils. The old-fashioned theories of partial differential equations proved inadequate for this purpose because they were evolved with the view of forcing the equation to disgorge a solution—or at least the skeleton of one—and were often in the nature of artifices the scope of which was limited and power unreliable. That moderately complete knowledge of the essential nature of a solution which has now been gained, together with the modern weapons of mathematics, such as the tensor calculus, provide an equipment both necessary and effective for weaving new matter into the older fabric.

In order that an adequate account of the theory of partial differential equations, refined and recast in moulds of precision, should be always at hand, the author has devoted the first ten chapters (163 pages) of his book to developing this theory. With as great a generality as is possible without making the existence theorems somewhat repulsive, yet at the same time without imposing tiresome restrictions on the work he is leading up to, the author establishes the basic theory of characteristics, canonical transformations of systems of equations, contact transformations and the Pfaff problem. Though it was designed as a necessary preliminary to the second half of the book, and indeed serves that end to perfection, it is actually an account remarkable in itself for lucidity and completeness.

Apart from an introductory chapter on maxima and minima in the ordinary sense, the second part of the book is devoted to founding the theory of the calculus of variations on the basis that has been mentioned. But though the basic association with partial differential equations is emphasised throughout, the link is not stressed to breaking point. Except in the examples by which the author illuminates the theory and puts its powers to the test, everything is stated in terms of many-dimensional space. Incidentally, the examples—they have an index of thirty entries to themselves—are extraordinarily interesting, and made all the more valuable by some clear and striking diagrams. Naturally the limits of space imposed upon the author have restricted the scope of the work, excluding discontinuous solutions and other topics to which the contributions of Carathéodory himself have been considerable. But the essence is there—in places perhaps a little concentrated, but never turbid. In brief, the author treats of the simple and the parametric variation problems, the second variation, the boundary problem, closed extremals and the Lagrange problem. An extensive bibliography is accompanied by ample indications to sources of information about those topics which are not contained in the text. In sum, this book is indispensable. E. L. I.