Long Sight

I WAS at school at Rossall, between Fleetwood and Blackpool, on the coast of Lancashire. One day, being on the seawall with Arthur A. Dawson, an Irish boy, we could see the Isle of Man as if it were ten miles away, and then to the south of the Calf of Man we could distinctly see on the horizon the summits of two mountains, which we pronounced must be in Ireland. Four years later I was staying at Blackpool with my mother, when we distinctly saw the same blue mountains just appearing above the sea. Being in the Isle of Man later on, I was at Port Erie, to the west of Castletown, and saw the same summits, and was told they were the mountains of Mourne. From there the mountains stood well out of the water, though we could not see the rest of the coast. The Mourne Mountains are 2798 feet high. They are 125 miles from Blackpool.

A. SHAW PAGE

Selsby Vicarage, Gloucestershire, May 28

Museums

THE interest which the readers of NATURE in this country and in America take in the promotion of museums has induced so many of them to inquire of me for a paper recently noticed by yourself that, to spare their time and my own, I shall be glad if you will enable me to refer inquirers to your advertising columns.

THE AUTHOR OF "MUSEUMS OF NATURAL HISTORY"

A NEW EXAMPLE OF THE USE OF THE INFINITE AND IMAGINARY IN THE SERVICE OF THE FINITE AND REAL

GEOMETERS are wont to speak (it seems to me) somewhat laxly of "the line at infinity" as if there were only one such line in a plane; in a certain but not in the most obvious sense this is true—viz. there is but one right line of which all the points are at an infinite distance from all lines external to them in the finite region of the plane, and except these points there are none others having this property; but in the sense that there is but one line infinitely distant from all points external to it in the finite region, the statement is obviously erroneous, for it need only to be mentioned to be at once perceived to be true by any tyro in geometry that all rays passing through either of the two "circular points at infinity" (Cayley's absolute) are infinitely distant from any external point in the finite region; these two imaginary points may indeed without any reference to the circle be *defined* as the points which radiate out in all directions rays infinitely distant from the finite region; the "absolute" being, so to say, the common depository, *i.e.* the crossing points of all infinitely distant rays as the "line at infinity" is the locus of all infinitely distant points. Similarly in space: there is not one infinitely distant plane, "the plane at infinity," but an infinitely infinite number of such planes—viz. any plane touching "the circle at infinity" (an imaginary circle in the plane at infinity) will at once be recognised to be infinitely distant from any external point in the finite region, or, as we may say more briefly and picturesquely, infinitely distant from the finite region itself. It will give greater vivacity to this conception to imagine an axis through which pass planes in all directions, and to travel in idea this axis round "the circle at infinity" keeping it always tangential thereto; the complex or corolla of planes, so to say, thus formed (infinitely infinite in number) contains only planes of infinite distance from the finite region; and "the plane at infinity" is but one of them—viz. the one which passes through all the axes named, just as the line at infinity in a plane is the line which passes through both the centres of infinite distance. The infinitely infinite series of infinitely distant planes is of course the correlative of the infinitely infinite series of infinitely distant points whose locus is the so-called "plane at infinity."

The above statements have only to be made, to be accepted by the geometer, although I do not remember seeing them * anywhere explicitly given; but what I want to show is that, although supersensuous abstractions, so far from being barren they are capable of immediate application to the world of reality, and afford an instantaneous answer to a very simple practical question which has only just lately been mooted. The question is this: Suppose $abcd$ to be a given pyramid, and that perpendiculars are drawn from its four vertices, say A, B, C, D, to a variable plane, then it is easy to show that a certain homogeneous quadratic function of A, B, C, D depending on the form of the pyramid or relative lengths of its edges must be constant, and the question arises, What is this constant quadratic function, this quadric in A, B, C, D, expressed in terms of the edges of the pyramid exclusively? † Just so if we take a triangle, abc, in a plane there will be a constant quadratic homogeneous function of the distances of its three vertices from a variable line; and it is well known in this case that if A, B, C are the distances the constant quadratic function in question will be—

$$(ab)^2(A-C)(B-C)+(bc)^2(B-A)(C-A)+ \\ (ca)^2(C-B)(AB).$$

But if we had not known this fact it could have been found as follows:—Calling the above function F, when A, B, C all or two of them become infinite the relation between the ratios of $A:B:C$ will be such as would arise from making $F=0$; on no other supposition will this be the case.

Now, if we use trilinear co-ordinates with abc as the triangle of reference, and take as the co-ordinates of any variable point P, the areas aPb, bPc, cPa instead of the simple distances of P from ab, bc, ca, then every body knows that the line at infinity has for its equation—

$$(1) \quad x+y+z=0,$$

and will easily see that the circle circumscribing abc has for its equation—

$$(2) \quad (ab)^2xy+(bc)^2(yz)+(ca)^2xz=0.$$

Moreover, when such co-ordinates are employed the distances of any line $Ax+By+Cz=0$ (3) from the three vertices are A, B, C each multiplied by the *same* known quantity.

If then A, B, C become infinite this line must pass through one of the intersections of the line at infinity with the circle, or, in other words, the equations (1), (3), (2) must be capable of being satisfied simultaneously, and accordingly by a well-known algebraical law it follows that the determinant to (2) *bordered* by the coefficients of (1) and (3) must vanish. Consequently this determinant so bordered will represent the sought-for form F, *i.e.* the constant quadratic function will be represented by—

$$\begin{array}{ccccc} \bullet & A & B & C & \bullet \\ A & \bullet & (ab)^2 & (ac)^2 & 1 \\ B & (ba)^2 & \bullet & (bc)^2 & 1 \\ C & (ca^2) & (cb)^2 & \bullet & 1 \\ \bullet & 1 & 1 & 1 & \end{array}$$

On calculating this determinant it will be found to be the function of $A-B$, $B-C$, $C-A$ above given, except that each term is multiplied by the constant factor -2, which may of course be dispensed with.

Now let us apply similar or analogous considerations to the determination of the constant quadratic function of

* The statement concerning the circular point-pair at infinity being centres of pencils of infinitely distant rays I have since met with somewhere in Dr. Salmon's Conics, but stated in quite a casual manner. It may not be unworthy of notice that just as the distance between any two points in a ray passing through either point of the *absolute* in a plane vanishes, so similarly vanishes the area of any triangle drawn in any plane touching "the imaginary circle at infinity" in space.

† If l, m, n, p are the distances of the vertices from the opposite faces, and x, y, z, t from the variable plane, it is well known that

$$\Sigma\left(\frac{x^2}{l^2}-2\,\frac{yz}{mn}\cos m, n\right)$$

is constant, in fact is unity.