A PLEA FOR THE MATHEMATICIAN

II.

I MIGHT go on, were it necessary, piling instance upon instance to prove the paramount importance of the faculty of observation to the process of mathematical discovery.* Were it not unbecoming to dilate on one's personal experience, I could tell a story of almost romantic interest about my own latest researches in a field where Geometry, Algebra, and the Theory of Numbers melt in a surprising manner into one another, like sunset tints or the colours of the dying dolphin, "the last still loveliest" (a sketch of which has just appeared in the Proceedings of the London Mathematical Society),† which would very strikingly illustrate how much observation, divination, induction, experimental trial, and verification—causation, too (if that means, as, if it mean anything, I suppose it must, mounting from phenomena to their reasons or causes of being)—have to do with the work of the mathematician. In the face of these facts, which every analyst can vouch for out of his own knowledge and personal experience, how can it be maintained, in the words of Professor Huxley (who, in this instance, is speaking of the sciences as they are in themselves and without any reference to scholastic discipline), that Mathematics "is that study which knows nothing of observation, nothing of induction, nothing of experiment, nothing of causation"?‡

I, of course, am not so absurd as to contend that the habit of observation of external nature‖ will be best or at all cultivated by the study of mathematics, leastways as that study is at present conducted; and no one can desire more earnestly than myself to see natural and experimental science introduced into our schools as a primary and indispensable branch of education: I think that that study and mathematical culture should go on hand in hand together, and that they would greatly influence each other for their mutual good. I should rejoice to see mathematics taught with that life and animation which the presence and example of her young and buoyant sister could not fail to impart; short roads preferred to long ones; Euclid honourably shelved or buried "deeper than did ever plummet sound" out of the schoolboy's reach; morphology introduced into the elements of Algebra; projection, correlation, and motion accepted as aids to geometry; the mind of the student quickened and elevated and his faith awakened by early initiation into the ruling ideas of polarity, continuity, infinity, and familiarisation with the doctrine of the imaginary and inconceivable.

It is this living interest in the subject which is so wanting in our traditional and mediæval modes of teaching. In France, Germany, and Italy, everywhere where I have been on the Continent, mind acts direct on mind in a manner unknown to the frozen formality of our academic institutions; schools of thought and centres of real intellectual co-operation exist; the relation of master and pupil is acknowledged as a spiritual and a lifelong tie connecting successive generations of great thinkers in an unbroken chain, just as we read, in the catalogue of our French Exhibition, or of the Salon at Paris, of this man or that being the pupil of one great painter or sculptor and the master of another. When followed out in this spirit, there is no study in the world which brings into more harmonious action all the faculties of the mind than the one of which I stand here as the humble representative and advocate. There is none other which prepares so many agreeable surprises for its followers, more wonderful than the transformation scene of a pantomime, or, like this, seems to raise them, by successive steps of initiation, to higher and higher states of conscious intellectual being.

This accounts, I believe, in part for the extraordinary longevity of all the greatest masters of the Analytical art, the Dii Majores of the mathematical Pantheon. Leibnitz lived to the age of 70; Euler to 76; Lagrange to 77; Laplace to 78; Gauss to 78; Plato, the supposed inventor of the conic sections, who made mathematics his study and delight, who called them the handles or aids to philosophy, the medicine of the soul, and is said never to have let a day go by without inventing some new theorems, lived to 82; Newton, the crown and glory of his race, to 85; Archimedes, the nearest akin, probably, to Newton in genius, to 75, and might have lived on to be 100, for aught we can guess to the contrary, when he was slain by the impatient and ill-mannered sergeant sent to bring him before the Roman General, in the full vigour of his faculties, and in the very act of working out a problem; Pythagoras, in whose school, I believe, the word mathematician (used, however, in a somewhat wider than its present sense) originated, the second founder of geometry, the inventor of the matchless theorem which goes by his name, the precognizer of undoubtedly the miscalled Copernican theory, the discoverer of the regular solids and the musical canon (who stands at the very apex of this pyramid of fame), if we may accept the tradition, after spending 22 years studying in Egypt and 12 in Babylon, opened school when 56 or 57 years old in Magna Græcia, married a young wife when past 60, and died, carrying on his work with energy unspent to the last, at the age of 99. The mathematician lives long and lives young; "the wings of his soul do not early drop off, nor do its pores become clogged with the earthy particles blown from the dusty highways of vulgar life."

Some people have been found to regard all mathematics, after the 47th proposition of Euclid, as a sort of morbid secretion, to be compared only with the pearl said to be generated in the diseased oyster, or, as I have heard it described, "une excroissance maladive de l'esprit humain,

* Newton's Rule (subsequently and for the first time deduced to demonstration in No. 2 of the London Mathematical Society's Proceedings) was to all appearance, and according to the more received opinion, obtained inductively by its author. So also my reduction of Euler's problem of the Virgins (or rather one slightly more general than this) to the form of a question (or, to speak more exactly, a set of questions) in simple partitions was (strangely enough) first obtained by myself inductively, the result communicated to Prof. Cayley, and proved subsequently by each of us independently, and by perfectly distinct methods.

† Under the title of "Outline Trace of the Theory of Reducible Cyclodes."

‡ Induction and analogy are the special characteristics of modern mathematics, in which theorems have given place to theories and no truth is regarded otherwise than as a link in an infinite chain. "Omne exit in infinitum" is their favourite motto and accepted axiom. No mathematician now-a-days sets any store on the discovery of isolated theorems, except as affording hints of an unsuspected new sphere of thought, like meteorites detached from some undiscovered planetary orb of speculation. The form, as well as matter, of mathematical science, as must be the case in any true living organic science, is in a constant state of flux and the position of its centre of gravity is liable to continual change. At different periods in its history, defined with more or less accuracy, as the science of number or quantity, or extension or operation or arrangement, it appears, at present, to be passing through a phase in which the development of the notion of continuity plays the leading part. In exemplification of the generalising tendency of modern mathematics, take so simple a fact as that of two straight lines or two planes being incapable of including "a space." When analysed this statement will be found to resolve itself into the assertion that if two out of the four triads that can be formed with four points lie respectively *in directum*, the same must be true of the remaining two triads; and that if two of the five tetrads that can be formed with five points lie respectively *in plano*, the remaining three tetrads (subject to a certain obvious exception) must each do the same. This at least is one way of arriving at the notion of an unlimited rectilinear and planar scheme of points. The two statements above made, translated into the language of determinants, immediately suggest as their generalised expression my great "Homaloidal Law," which affirms that the vanishing of a certain specifiable number of minor determinants of a given order of any matrix (*i.e.* rectangular array of quantities) implies the simultaneous evanescence of all the rest of that order. I made (*inter alia*) a beautiful application of this law (which is, I believe, recorded in Mr. Spottiswoode's valuable treatise on Determinants, but where besides I know not) to the establishment of the well-known relations, wrung out with so much difficulty by Euler, between the cosines of the nine angles which two sets of rectangular ones in space make with one another. This is done by contriving a matrix such that the six known equations connecting the nine cosines taken both ways in sets of threes shall be expressed by the evanescence of six of its minors; the simultaneous evanescence of the remaining minors given by the Homaloidal Law will then be found to express the Eulerian relations in question, which are thus obtained by a simple process of inspection and reading off, without any labour whatever. The fact that such a law, containing in a latent form so much refined algebra, and capable of such interesting immediate applications, should present itself to the *observation* merely as the extended expression of the ground of the possibility of our most elementary and seemingly intuitive conceptions concerning the right line and plane, has often filled me with amazement to think of.

‖ As the prerogative of Natural Science is to cultivate a taste for observation, so that of Mathematics is, almost from the starting point, to stimulate the faculty of invention.

Others find its justification, its "raison d'être," in its being either the torch-bearer leading the way, or the handmaiden holding up the train of Physical Science; and a very clever writer in a recent magazine article, expresses his doubts whether it is, in itself, a more serious pursuit, or more worthy of interesting an intellectual human being, than the study of chess problems or Chinese puzzles.* What is it to us, they say, if the three angles of a triangle are equal to two right angles, or if every even number is, or may be, the sum of two primes,† or if every equation of an odd degree must have a real root? How dull, stale, flat and unprofitable are such and such like announcements! Much more interesting to read an account of a marriage in high life, or the details of an international boat-race. But this is like judging of architecture from being shown some of the brick and mortar, or even a quarried stone of a public building—or of painting from the colours mixed on the palette, or of music by listening to the thin and screechy sounds produced by a bow passed haphazard over the strings of a violin. The world of ideas which it discloses or illuminates, the contemplation of divine beauty and order which it induces, the harmonious connexion of its parts, the infinite hierarchy and absolute evidence of the truths with which mathematical science is concerned, these, and such like, are the surest grounds of its title to human regard, and would remain unimpaired were the plan of the universe unrolled like a map at our feet, and the mind of man qualified to take in the whole scheme of creation at a glance.

In conformity with general usage, I have used the word mathematics in the plural; but I think it would be desirable that this form of word should be reserved for the applications of the science, and that we should use mathematic in the singular number to denote the science itself, in the same way as we speak of logic, rhetoric, or (own sister to algebra‡) music. Time was when all the parts of the subject were dissevered, when algebra, geometry, and arithmetic either lived apart or kept up cold relations of acquaintance confined to occasional calls upon one another; but that is now at an end; they are drawn together and are constantly becoming more and more intimately related and connected by a thousand fresh ties, and we may confidently look forward to a time when they shall form but one body with one soul. Geometry formerly was the chief borrower from arithmetic and algebra, but it has since repaid its obligations with overflowing usury; and if I were asked to name, in one word, the pole-star round which the mathematical firmament revolves, the central idea which pervades as a hidden spirit the whole corpus of mathematical doctrine, I should point to Continuity as contained in our notions of space, and say, It is this, it is this! Space is the Grand *Continuum* from which, as from an inexhaustible reservoir, all the fertilizing ideas of modern analysis are derived; and as Brindley, the engineer, once allowed before a parliamentary committee that, in his opinion, rivers were made to feed navigable canals, I feel sometimes almost tempted to say that one principal reason for the existence of space, or at least one principal function which it discharges, is that of feeding mathematical invention. Everybody knows what a wonderful influence geometry has exercised in the hands of Cauchy, Puiseux, Riemann, and his followers Clebsch, Gordan, and others, over the very form and presentment of the modern calculus, and how it has come to pass that the tracing of curves, which was once to be regarded as a puerile amusement, or at best useful only to the architect or decorator, is now entitled to take rank as a high philosophical exercise, inasmuch as every new curve or surface, or other circumspection of space, is capable of being regarded as the synthesis and embodiment of some specific organised system of continuity.*

The early study of Euclid made me a hater of geometry, which I hope may plead my excuse if I have shocked the opinions of any in this room (and I know there are some who rank Euclid as second in sacredness to the Bible alone, and as one of the advanced outposts of the British Constitution) by the tone in which I have previously alluded to it as a school-book; and yet, in spite of this repugnance, which had become a second nature in me whenever I went far enough into any mathematical question, I found I touched, at last, a geometrical bottom; so it was, I may instance, in the purely arithmetrical theory of partitions; so, again, in one of my more recent studies the purely algebraical question of the invariantive criteria of the nature of the roots of an equation of the fifth degree; —the first inquiry landed me in a new theory of polyhedra, the latter found its perfect and only possible complete† solution in the construction of a surface of the ninth order and the sub-division of its infinite contents into three distinct natural regions.‡

Having thus expressed myself at greater length

* Is it not the same disregard of principles, the same *indifference to truth for its own sake*, which prompts the question "Where's the good of it?" in reference to speculative science, and "Where's the harm of it?" in reference to white lies and pious frauds? In my own experience I have found that the very same people who delight to put the first question are in the habit of acting upon the denial implied in the second. *Abit in mores incuria.*

† This theorem still awaits proof; it is stated, I believe, in Euler's correspondence with Goldbach: I re-discovered it in ignorance of Euler's having mentioned it, in connection with a theory of my own concerning cubic forms. The evidence in its favour is *induction* of the undemonstrative or purely accumulative kind, and it may or may not turn out eventually to be true. As a most learned scholar who heard this address given at Exeter remarked to me not many days ago, it is certainly by no process of deduction that we make out that five times six is thirty. I mention this, because I know some, who agree, or did agree, with Professor Huxley's published opinions about mathematics, are under the impression that the higher processes of mind in mathematics only concern "the aristocracy of mathematicians:" on the contrary, they lie at the very foundations of the subject. There are besides, and in abundance, mathematical processes which only by a forced interpretation can be brought under the head of demonstration, whether deductive or inductive, and really belong to a sort of artistic and constructive faculty, such for example as evaluating definite integrals, or making out the best way one can the number of distinct branches, and the general character of each branch of a curve from its algebraical equation.

‡ I have elsewhere (in my Trilogy published in the "Philosophical Transactions") referred to the close connection between these two cultures, not merely as having Arithmetic for their common parent, but as similar in their habits and affections. I have called "Music the Algebra of sense, Algebra the Music of the reason; Music the dream, Algebra the waking life—the soul of each the same!"

* M. Camille Jordan's application of Dr. Salmon's Eikosi-heptagram to Abelian functions is one of the most recent instances of this reverse action of geometry on analysis. Mr. Crofton's admirable apparatus of a reticulation with infinitely fine meshes rotated successively through indefinitely small angles, which he applies to obtaining whole families of definite integrals, is another equally striking example of the same phenomenon.

† *Complete* in the sense of *universal*, more *than perfect or complete* in the ordinary sense. Two criteria are absolutely fixed; but in addition to these two an additional criterion or set of criteria must be introduced to make the system of conditions sufficient. The number of such set may be either one or whatever number we please, and into such one or into each of the set (if more than one) an indefinite number of arbitrary parameters (limited) may be introduced. Now the geometrical construction I arrive at contains implicitly the totality of all these infinitely varied forms of criteria, or sets of criteria, and without it, the existence and possibility of such variety in the shape of the solution could never have been anticipated or understood. My truly eminent friend M. Charles Hermite (Membre de l'Institut), with all the efforts of his extraordinary analytical power, and with the knowledge of my results to guide him, has only been able by the non-geometrical method to arrive at one form of solution consisting of a third criterion absolutely definite and destitute of a single variable parameter. As is well known, I have made a very important use of a criterion of the same form as M. Hermite's, but containing one arbitrary parameter (limited). The subject will be found resumed from the point where I left it, and pursued in considerable detail by Prof. Cayley, in one of his more recent memoirs on Quartics in the Philosophical Transactions. M. Hermite it was who first surprised Invariantists (l'Eglise Invariantive, as we are sometimes styled) by an *à priori* demonstration that the nature of the roots or factors of quartics could in general be found by means of invariantive criteria. This was known to be possible up to the *fourth* order of binary quartics, and impossible for the *fourth*. M. Hermite showed that this negation which seemed to stop the way to further progress was an exceptional case; that whereas for the second, third, fifth, sixth, and all higher degrees the thing could be done, for the fourth alone it was impossible: as regards linear quartics, the question does not arise. I look upon this failure of a law for one term in the middle of an infinite progression as an unparalleled *miracle of arithmetic*, far more real and deeper seated than the one alluded to by Mr. Babbage in connection with the discontinuous action of a supposed machine in his ninth Bridgwater Treatise.

‡ So I found, as a pure matter of observation, that allineation (*alignement*) in ornamental gardening—*i.e.* the method of putting trees in positions to form a very great number or the greatest number possible of straight rows, of which a few special cases only had been previously considered as detached porismatic problems, forms part of a great connected theory of the pluperfect points on a cubic curve, those points, of which the nine points of inflection and Plücker's twenty-seven points serve as the lowest instances.

than I originally intended on the subject, which, as standing first on the muster roll of the Association, and as having been so recently and repeatedly arraigned before the bar of public opinion, is entitled to be heard in its defence (if anywhere) in this place,—having endeavoured to show what it is not, what it is, and what it is probably destined to become, I feel that I must enough and more than enough have trespassed on your forbearance.

J. J. SYLVESTER

The remarks on the use of experimental methods in mathematical investigation led to Dr. Jacobi, the eminent physicist of St. Petersburg, who was present at the delivery of the foregoing address, favouring me with the annexed anecdote relative to his illustrious brother, C. G. J. Jacobi*—

"En causant un jour avec mon frère défunt sur la nécessité de contrôler par des expériences réitérées toute observation, même si elle confirme l'hypothèse, il me raconta avoir découvert un jour une loi très-remarquable de la théorie des nombres, dont il ne douta guère qu'elle fût générale. Cependant par un excès de précaution ou plutôt pour faire le superflu, il voulut substituer un chiffre quelconque réel aux termes généraux, chiffre qu'il choisit au hasard, ou, peut-être, par une espèce de divination, car en effet ce chiffre mit sa formule en défaut ; tout autre chiffre qu'il essaya en confirma la généralité. Plus tard il réussit à prouver que le chiffre choisi par lui par hasard, appartenait à un système de chiffres qui faisait la seule exception à la règle.

"Ce fait curieux m'est resté dans la mémoire, mais comme il s'est passé il y a plus d'une trentaine d'années, je ne rappelle plus les détails.

"M. H. JACOBI

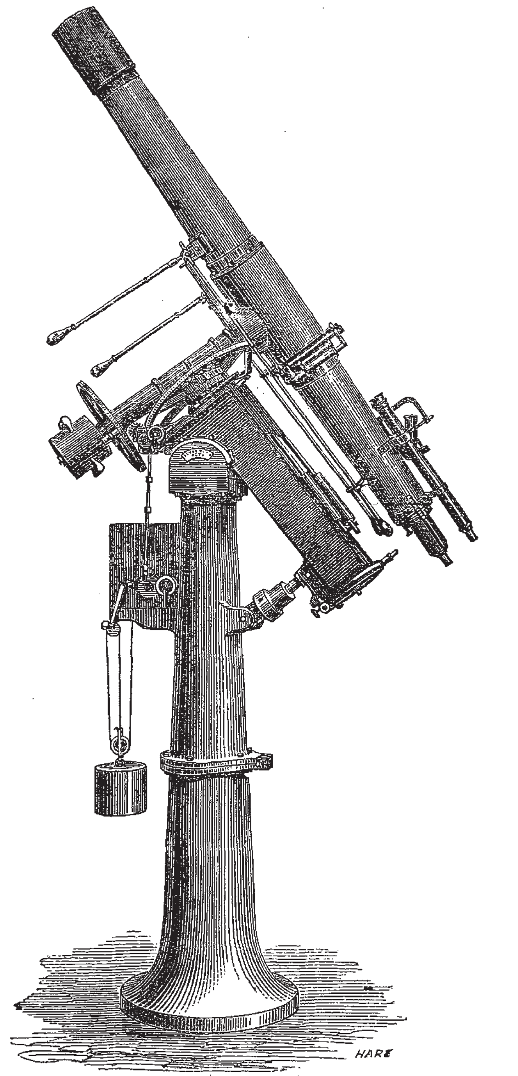
"Exeter, 24 Août, 1869."

THE NEW TELESCOPE AT ETON

IN furtherance of natural science work at Eton, an excellent telescope has been recently given to the school by the energy and liberality of some of the masters.

The instrument is a refractor, with object glass of 5·9 inches clear aperture, and 88 inches focus, and was made by Messrs. Cooke and Sons, of York, who also supplied the observatory and superintended the erection of the telescope. It is, as will be seen from the engraving, mounted equatorially on the German system, with declination circle reading to 10″ of arc, and hour circle reading to 2″ of time. The mechanical details do not, with one exception, deviate materially from the pattern usually adopted by Messrs. Cooke, whose name is a guarantee for skill of design and excellence of workmanship. The exception alluded to is in the construction of the driving clock, the speed of which is not regulated, as usual, by a centrifugal governor, or fly, alone, but by a fly supplemented by an ordinary clock escapement. This arrangement is quite new, and is the invention of the late Mr. T. Cooke, the senior partner in the firm. It was described by him in a paper read before the Royal Astronomical Society a short time ago. The details would hardly be intelligible without drawings, but the general mode of action is as follows :—

The barrel is connected with two trains of wheel-work : one (the lowest wheel of which gives motion in the ordinary way to the telescope) is terminated by a fly of insufficient power *per se* to reduce the speed within proper limits ; the other train is terminated by a half-dead escapement of the usual kind. One of the wheels of the fly-train has a broad rim, on which presses a brake actuated by a wheel in the escapement train. When the escapement is stopped, this brake presses on the wheel with sufficient force to stop the motion of the clock entirely. When the escapement is set to work the brake is released, and the fly-train moves, communicating motion to the telescope. If the speed becomes too great, so as to outrun the escapement, the latter immediately applies increased brake-power, and checks the motion of



THE ETON EQUATORIAL

the fly ; and *vice versâ*, if from increased friction or other cause the motion is too slow, so that the fly lags behind the escapement, the brake-spring is relaxed by the latter until the due speed is regained. Thus the two trains are balanced against each other, and since one of the wheels of the escapement-train is, as in some forms of train *remontoires*, supported in a swinging-frame (which frame, in fact, controls the brake-spring), the intermittent motion of the escapement does not reach the telescope. This clock seems to work very smoothly; and not the least advantage of the arrangement is the facility with which

* It is said of Jacobi, that he attracted the particular attention and friendship of Böckh, the director of the philological seminary at Berlin, by the zeal and talent he displayed for philology, and only at the end of two years' study at the University, and after a severe mental struggle, was able to make his final choice in favour of mathematics. The relation between these two sciences is not perhaps so remote as may at first sight appear ; and indeed it has often struck me that metamorphosis runs like a golden thread through the most diverse branches of modern intellectual culture, and forms a natural link of connection between subjects in their aims so remote as grammar, philology, ethnology, rational mythology, chemistry, botany, comparative anatomy, physiology, physics, algebra, versification, music, all of which, under the modern point of view, may be regarded as having morphology for their common centre. Even singing, I have been told, the advanced German theorists regard as being strictly a development of recitative, and infer therefrom that no essentially new melodic themes can be invented until a social cataclysm, or the civilisation of some at present barbaric races, shall have created fresh necessities of expression, and called into activity new forms of impassioned declamation.